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Computing the perturbative gluon condensate

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Abstract: The normalization of the gluon condensate and of renormalon-related power corrections in QCD is computed under the assumption that their "perturbative" part dominates over any eventual extra contribution from the non-trivial vacuum. The calculation is performed in the infrared finite coupling framework, assuming an infrared fixed point is present in the perturbative coupling down to low values of N_f. The freezing perturbative coupling is reconstructed using a Banks-Zaks expansion approach. Parameter-free predictions of the low energy moments of the coupling, which determine the process-independent part of the power corrections, are obtained for a number of choices of the running coupling.

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Contents

1. Introduction

Recent years have witnessed a rapid development of the area covering the borderline between "perturbative" and "non-perturbative" physics in QCD. In particular, perturbative ideas have been pushed increasingly far towards the low-energy frontier to deal with the phenomenology of power corrections. Still in these advances the normalization of power corrections is usually considered as an incalculable "non-perturbative" parameter, to be fitted from the data. This has been the situation ever since their original introduction [1] by Shifman, Vainshtein and Zakharov (SVZ). In this note, I suggest a possibility to compute these parameters from first principles, for the limited class of "renormalon-related" power corrections which include in particular the "gluon condensate". Among the various methods [2] devised to deal with these contributions, the "infrared finite coupling" approach [3, 4, 5] stands out as a particularly attractive scheme. A framework where this approach can be justified has recently been suggested [6], and the possibility of a calculation of power corrections from *perturbative* input has been pointed out. The aim of this paper is to implement the latter suggestion. After a brief review (section 2) of the infrared (IR) finite coupling approach and of the proposal in [6], a method to construct the IR finite coupling from the Banks-Zaks expansion is described in section 3. The results for the power corrections are given in section 4, and further discussed in section 5 which also contains the conclusions.

2. A framework for the IR finite coupling approach to power corrections

In this approach the power corrections are parametrized in term of low energy moments of a "non-perturbative" coupling $\bar{a} = a + \delta a$ assumed to be IR finite, where a ($a \equiv \alpha_s/\pi$) is the "perturbative part", and δa a "non-perturbative" modification needed to make $\bar{a}$ IR finite. Although the approach can also deal with minkowskian quantities, consider as a simple example the case of an euclidean observable $D(Q^2)$ in the "single dressed gluon exchange approximation" [7, 8]

$$D(Q^2) = \int_0^\infty \frac{dk^2}{k^2}\ \bar{a}(k^2)\ \Phi\left(\frac{k^2}{Q^2}\right), \tag{2.1}$$

where $\Phi(k^2/Q^2)$, the "loop momentum distribution function" [2], is known [9] from the relevant single dressed gluon diagrams. Introducing an IR cut-off μ_I to separate long and short distances, the right hand side is approximated at large Q^2 by

$$D(Q^2) \simeq c_n \frac{\lambda_n(\mu_I)}{Q^{2n}} + \int_{\mu_I^2}^\infty \frac{dk^2}{k^2} a(k^2)\Phi\left(\frac{k^2}{Q^2}\right), \tag{2.2}$$

with $\lambda_n(\mu_I) = \int_0^{\mu_I^2} \frac{dk^2}{k^2}\bar{a}(k^2)k^{2n}$, where I assumed that $\Phi(k^2/Q^2) \simeq c_n(k^2/Q^2)^n$ at low k^2 and the contribution of the δa piece has been neglected above μ_I. These steps can now be justified with the following two crucial assumptions:

i) The *perturbative* part a of the coupling reaches a non-trivial IR fixed point at low scales and is IR finite by itself, without the need for an hypothetical δa contribution. This statement is likely to be correct for N_f slightly below 16.5 where the perturbative coupling has a Banks-Zaks fixed point [10, 11, 12] beyond one-loop, and I assume it is still true down to $N_f = 0$. This assumption is supported [13, 14, 15] by the behavior of the Banks-Zaks expansion for some QCD effective charges.

Actually, the previous statement must be correct within a range $N_f^* < N_f < 16.5$ which defines the "conformal window" where the perturbative coupling is IR finite and causal [16, 6]. Within the conformal window, there is by definition no δa term, and we have

$$D(Q^2) = D_{\overline{PT}}(Q^2)\,, \tag{2.3}$$

with

$$D_{\overline{PT}}(Q^2) \equiv \int_0^\infty \frac{dk^2}{k^2} a(k^2)\Phi\left(\frac{k^2}{Q^2}\right). \tag{2.4}$$

At large Q^2, one obtains as in eq. (2.2)

$$D_{\overline{PT}}(Q^2) \simeq \frac{C_{PT}(\mu_I)}{Q^{2n}} + \int_{\mu_I^2}^\infty \frac{dk^2}{k^2} a(k^2)\Phi\left(\frac{k^2}{Q^2}\right), \tag{2.5}$$

with the normalization of the power correction

$$C_{PT}(\mu_I) = c_n \int_0^{\mu_I^2} \frac{dk^2}{k^2} a(k^2) k^{2n} \tag{2.6}$$

given by a low energy moment of the *perturbative* coupling. Since the latter is no more causal below N_f^* (even if it still IR finite there), eq. (2.3) cannot be correct anymore for $N_f < N_f^*$, where the "conformal window amplitude" $D_{\overline{PT}}(Q^2)$ is expected to have unphysical Landau singularities in the (complex) Q^2 plane. We must therefore have

$$D(Q^2) = D_{\overline{PT}}(Q^2) + D_{\overline{NP}}(Q^2)\,, \tag{2.7}$$

where[1] the "genuine non-perturbative piece" $D_{\overline{NP}}(Q^2)$ cancels the Landau singularities present in $D_{\overline{PT}}(Q^2)$. In the standard IR finite coupling approach this piece would correspond to the contribution of the δa part of the coupling in eq. (2.1). Since the existence of such a term is quite hypothetical, I shall *not* assume that the $D_{\overline{NP}}(Q^2)$ piece is related to a (universal) non-perturbative QCD coupling. Still at large Q^2 this piece may contribute a "non-perturbative component" C_{NP} to the $\mathcal{O}(1/Q^{2n})$ power correction

$$D_{\overline{NP}}(Q^2) \simeq \frac{C_{NP}}{Q^{2n}} \tag{2.8}$$

so that below N_f^* we have

$$D(Q^2) \simeq D_{\overline{PT}}(Q^2) + \frac{C_{NP}}{Q^{2n}} \tag{2.9}$$

hence

$$D(Q^2) \simeq \frac{C(\mu_I)}{Q^{2n}} + \int_{\mu_I^2}^{\infty} \frac{dk^2}{k^2} a(k^2) \Phi\left(\frac{k^2}{Q^2}\right), \tag{2.10}$$

with

$$C(\mu_I) = C_{PT}(\mu_I) + C_{NP}\,. \tag{2.11}$$

ii) The second crucial assumption I shall make is that the "non-perturbative" component C_{NP} can in fact be neglected (for not too small μ_I) in eq. (2.11). This assumption, which actually takes the exact counterpart of the SVZ hypothesis [1] that the "genuine non-perturbative piece" C_{NP} dominates over the "perturbative" fluctuations, can be justified [6] in a number of ways. One is to observe that C_{NP}, which vanishes identically for $N_f^* < N_f < 16.5$ within the conformal window, may still be small for $N_f < N_f^*$ below the conformal window, provided N_f is close enough to N_f^*. In [16, 6], it was found that in fact $4 < N_f^* < 6$, which makes it at least plausi-

[1] If $D_{\overline{PT}}(Q^2)$ is interpreted as the analytic continuation in N_f of the full conformal window amplitude, the decomposition eq. (2.7) is general [16, 6] and valid beyond the single dressed gluon approximation of eq. (2.1).

ble the neglect of C_{NP} at the "real life" QCD value $N_f = 3$. Another (more drastic) possibility is that the power corrections in $D_{\overline{PT}}(Q^2)$ and $D_{\overline{NP}}(Q^2)$ do not match (even though at low Q^2 the two components cancel their mutual Landau singularities below the conformal window), i.e. that the power corrections are either entirely "perturbative" and contribute only to $D_{\overline{PT}}(Q^2)$, or entirely "non-perturbative" and contribute only to $D_{\overline{NP}}(Q^2)$. This would mean that $C_{NP} \equiv 0$ even *below* N_f^*, and only the C_{PT} component is present, for those condensates (like the gluon condensate) which do not vanish within the conformal window, whereas $C_{NP} \neq 0$ below N_f^* only for those condensates (like the quark condensate) which vanish identically within the conformal window, and therefore have no C_{PT} component. In such a case, the neglect of C_{NP} would be justified at all N_f's for the "conformal window type" of power corrections. Anyway, the working hypothesis in the following shall be that one can compute the bulk of the latter type of power corrections from eq. (2.6) alone. In this way, the IR finite coupling approach not only finds a natural framework, but its predictiveness is enhanced since there is not any more any "non-perturbative" free parameter and the normalization of power corrections can be computed, as we demonstrate in the next section (in this sense the approach goes beyond the operator product expansion even when applied to euclidean quantities).

3. Reconstructing the IR finite perturbative coupling: a Banks-Zaks expansion approach

Even though the perturbative coupling appears to have an IR fixed point for large enough N_f beyond one-loop, this is not always manifest when one decreases N_f. For instance, the Banks-Zaks fixed point at two-loop relies on having $\beta_1 < 0$, which is not realized for $N_f < 8$. Then one might rescue the fixed point with a negative three-loop term, but even this feature is usually lost at $N_f = 3$. On the other hand, as mentioned in section 2, the Banks-Zaks expansion does signal in a number of cases the persistence of the fixed point even down to $N_f = 2$. This observation suggests the following strategy: try to reconstruct the IR finite coupling, and eventually supply the missing higher order terms in the beta function $\beta(a)$, given the IR fixed point Banks-Zaks expansion. This is an expansion in powers of the distance $16.5 - N_f$ from the top of the conformal window, which is proportional to β_0. The solution $a^* = a^*(\epsilon)$ of the equation

$$\beta(a) = \beta(a, \epsilon) = -\beta_0\ a^2 - \beta_1\ a^3 - \beta_2\ a^4 - \beta_3\ a^5 + \cdots = 0 \tag{3.1}$$

in the limit $\beta_0 \to 0$, with β_i $(i \geq 1)$ finite is obtained as a power series

$$a^* = a^*(\epsilon) = \epsilon + \delta_1\ \epsilon^2 + \delta_2\ \epsilon^3 + \cdots , \tag{3.2}$$

where

$$\begin{aligned}\delta_1 &= \beta_{1,1} - \frac{\beta_{2,0}}{\beta_{1,0}} \\ \delta_2 &= \delta_1^2 + g_1\ \delta_1 + \beta_{2,1} - g_2\,.\end{aligned} \tag{3.3}$$

The expansion parameter [12] is $\epsilon \equiv \frac{8}{321}(16.5 - N_f) = -\beta_0/\beta_{1,0}$. The $\beta_{i,j}$, which are N_f-independent (but scheme dependent for $i > 1$), are defined by $\beta_1 = \beta_{1,0} + \beta_{1,1}\,\beta_0$ ($\beta_{1,0} = -107/16$, $\beta_{1,1} = 19/4$), $\beta_2 = \beta_{2,0} + \beta_{2,1}\,\beta_0 + \beta_{2,2}\,\beta_0^2$ (I assume β_2 is at most quadratic in N_f, hence in β_0) and g_1, g_2 are given in eq. (3.6). Given the knowledge of the 3-loop beta function in (e.g.) the $\overline{MS}$ scheme, $\beta_{2,0}$ can be obtained [17] from a one-loop calculation of a (see eq. (4.2)). I shall also use the related expansion for the critical exponent

$$\gamma = \gamma(\epsilon) = \frac{\partial\beta}{\partial a}(a^*, \epsilon) \tag{3.4}$$

$$= -\beta_{1,0}\epsilon^2(1 + g_1\epsilon + g_2\epsilon^2 + \cdots)\,, \tag{3.5}$$

where

$$\begin{aligned}g_1 &= \beta_{1,1} \\ g_2 &= g_1^2 + \frac{\beta_{3,0}}{\beta_{1,0}} - \left(\frac{\beta_{2,0}}{\beta_{1,0}}\right)^2.\end{aligned} \tag{3.6}$$

The g_i's are scheme independent [12], and [13] $g_2 = -8.89$.

The method relies on the differential equation [12] for $a^*(\epsilon)$

$$\frac{\partial}{\partial\epsilon}[\epsilon\nu(a^*, \epsilon)] = \gamma(\epsilon)\frac{da^*}{d\epsilon}\,, \tag{3.7}$$

where $\nu(a, \epsilon)$ is the N_f dependent part of the beta function, after splitting off the $\epsilon = 0$ (i.e. $N_f = 16.5$) piece $\beta(a, 0)$

$$\beta(a, \epsilon) \equiv \beta(a, 0) - \epsilon\nu(a, \epsilon)\,. \tag{3.8}$$

Its expansion in powers of a is

$$\nu(a, \epsilon) = -\beta_{1,0}a^2[1 + \beta_{1,1}a + (\beta_{2,1} + \beta_{2,2}\beta_0)a^2 + \cdots]\,. \tag{3.9}$$

Eq. (3.7) follows by taking the total derivative with respect to ϵ of the relation $\beta(a^*, \epsilon) = 0$ which defines the fixed point $a^*(\epsilon)$

$$\frac{\partial\beta}{\partial a}(a^*, \epsilon)\frac{da^*}{d\epsilon} + \frac{\partial\beta}{\partial\epsilon}(a^*, \epsilon) = 0 \tag{3.10}$$

and using eq. (3.4) and $\frac{\partial\beta}{\partial\epsilon}(a, \epsilon) = -\frac{\partial}{\partial\epsilon}[\epsilon\nu(a, \epsilon)]$ (eq. (3.8)).

It is convenient to introduce the function $\epsilon^*(a)$, which is the inverse of the Banks-Zaks function $a^*(\epsilon)$: for given a, $\epsilon^*(a)$ is the value of ϵ (i.e. of N_f) where $\beta(a,\epsilon) = 0$. The knowledge of $\epsilon^*(a)$ and of $\nu(a,\epsilon)$ determine $\beta(a,0)$, hence the full beta function. Indeed using eq. (3.8) the condition $\beta[a,\epsilon^*(a)] = 0$ becomes

$$\beta(a,0) = \epsilon^*(a)\nu[a,\epsilon^*(a)]\,. \tag{3.11}$$

Hence

$$\beta(a,\epsilon) = \epsilon^*(a)\nu[a,\epsilon^*(a)] - \epsilon\;\nu(a,\epsilon)\,. \tag{3.12}$$

In term of $\epsilon^*(a)$ eq. (3.7) reads

$$\frac{\partial}{\partial\epsilon}[\epsilon\nu(a,\epsilon)]|_{\epsilon=\epsilon^*} = \frac{\gamma(\epsilon^*)}{d\epsilon^*/da}\,. \tag{3.13}$$

Eq. (3.13) gives a constraint on $\nu(a,\epsilon)$ given the Banks-Zaks functions $\gamma(\epsilon)$ and $a^*(\epsilon)$. This constraint is not sufficient to determine $\nu(a,\epsilon)$ (and the beta function) without further assumptions. In the following I shall assume that $\nu(a,\epsilon) = \nu_0(a)$ is independent of ϵ, i.e. that the beta function coefficients are at most linear in N_f (or β_0): this amounts to an approximation, in the spirit of the Banks-Zaks approach, where one keeps only the leading $\epsilon = 0$ term in an expansion of $\nu(a,\epsilon)$ in powers of ϵ (in particular, one neglects the $\beta_{2,2}\;\beta_0$ term in eq. (3.9)). Then eq. (3.13) gives

$$\nu_0(a) = \frac{\gamma[\epsilon^*(a)]}{d\epsilon^*/da} \tag{3.14}$$

and from eq. (3.12) one gets

$$\beta(a,\epsilon) = [\epsilon^*(a) - \epsilon]\frac{\gamma[\epsilon^*(a)]}{d\epsilon^*/da}\,. \tag{3.15}$$

Using the Banks-Zaks expansions of the fixed point $a^*(\epsilon)$ and of the critical exponent $\gamma(\epsilon)$ truncated to a given order as input, eq. (3.15) yields a corresponding "improved" approximation to the beta function, which displays a built-in fixed point at $a = a^*(\epsilon)$.

In this approach, the leading order (LO) approximation thus gives $\epsilon^*(a) = a$ and $\gamma[\epsilon^*(a)] = -\beta_{1,0}a^2$. The next-to-leading order (NLO) approximation uses the NLO Banks-Zaks expansions of the fixed point and of the critical exponent: $\epsilon^*(a)$ is then obtained by inverting eq. (3.2) (with $\delta_2 = 0$), i.e. solving for ϵ^* in $a = \epsilon^* + \delta_1\epsilon^{*2}$ and reporting in eq. (3.15), with $\gamma(\epsilon^*) = -\beta_{1,0}\epsilon^{*2}(1 + g_1\epsilon^*)$. The next-to-next-to-leading order (NNLO) approximation uses the NNLO Banks-Zaks expansions of the fixed point (which requires the knowledge of $\beta_{2,1}$) and of the critical exponent (eq. (3.2) and (3.5)), etc ...

The approximation can be further systematically improved by including the knowledge of the known N_f-dependent terms in the beta function. For instance, if the three-loop β_2 coefficient is known, one can include the knowledge[2] of the term quadratic in β_0 in β_2 with the ansatz

$$\nu(a,\epsilon) = \nu_0(a) + \beta_{1,0}^2 \beta_{2,2} a^4 \epsilon\,, \tag{3.16}$$

where $\nu_0(a) \equiv \nu(a,0)$ is independent of ϵ (the knowledge of $\beta_{2,0}$ and $\beta_{2,1}$ is contained in the NLO and NNLO terms in the Banks-Zaks expansion of a^*, as mentioned above). Eq. (3.13) then fixes $\nu_0(a)$ from

$$\nu_0(a) + 2\beta_{1,0}^2 \beta_{2,2} a^4 \epsilon^*(a) = \frac{\gamma[\epsilon^*(a)]}{d\epsilon^*/da}\,, \tag{3.17}$$

which yields $\nu(a,\epsilon)$, hence from eq. (3.12)

$$\beta(a,\epsilon) = [\epsilon^*(a) - \epsilon]\frac{\gamma[\epsilon^*(a)]}{d\epsilon^*/da} - \beta_{1,0}^2 \beta_{2,2} a^4 [\epsilon^*(a) - \epsilon]^2\,. \tag{3.18}$$

4. Results

The "coupling" appearing in eq. (2.4) should be viewed as a physical, gauge-independent quantity, just as the observable $D(Q^2)$ to which it is directly related. In the IR finite coupling approach, it is also assumed to be universal, i.e. the same for all observables. The existence of such an object is still speculative. It is attractive to identify this coupling to the "skeleton coupling" [7, 8, 18] associated to a (yet hypothetical) "QCD skeleton expansion". A promising approach in this direction is provided by the "pinch technique" construction [19, 20]. The pinch coupling is presently known only at one-loop, where it is related to the $\overline{MS}$ coupling by

$$a(k^2) = a_{\overline{MS}}(\mu^2) + \left[-\beta_0\left(\log\left(\frac{k^2}{\mu^2}\right) - \frac{5}{3}\right) + d_1\right] a_{\overline{MS}}^2(\mu^2) + \cdots\,, \tag{4.1}$$

with $d_1|\text{pinch} = 1$. An alternative suggestion [3] is to use the "gluon bremsstrahlung coupling" [21], also known to the one-loop level eq. (4.1) with $d_1|\text{brems} = 1 - \pi^2/4$. Since the full three-loop beta function coefficient (hence δ_2) is not yet known for these two couplings, I shall apply the method of section 3 in the NLO approximation described there. Actually, since the Banks-Zaks expansion of the critical exponent is known [13] up to NNLO (eq. (3.5)), and may be reliable [6] even down to $N_f = 3$, I shall go half-way towards the NNLO approximation, and use eq. (3.5) in eq. (3.15), while still using eq. (3.2) (with $\delta_2 = 0$) to fix $\epsilon^*(a)$. The input scheme dependent numerical values following from eq. (4.1) and the relation [17]

$$d_1 = -\frac{\beta_{2,0} - \beta_{2,0}^{\overline{MS}}}{\beta_{1,0}} \tag{4.2}$$

[2]Actually, given that $0 < a < a^* = \mathcal{O}(\epsilon)$, this term is effectively of the same order as the $\beta_{3,1}a^5$ term in $\nu(a,\epsilon)$ (eq. (3.9)), and should be taken as input only together with the latter, i.e. at the NNNLO level.

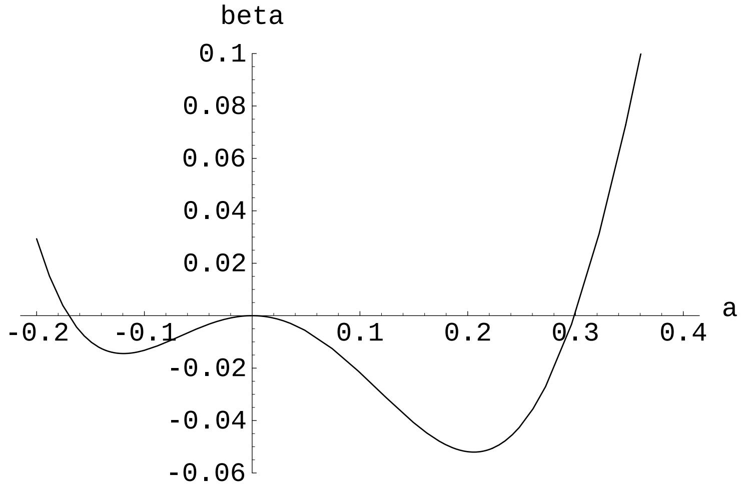

Figure 1: The bremsstrahlung coupling beta function ($N_f = 3$).

are [18] $\frac{\beta_{2,0}}{\beta_{1,0}}|\text{pinch} = 2.61$ and $\frac{\beta_{2,0}}{\beta_{1,0}}|\text{brems} = 2.61+\pi^2/4 = 5.08$. Hence $\delta_1|\text{pinch} = 2.14$ while $\delta_1|\text{brems} = -0.33$ (a smaller correction!). It follows from eq. (3.2) that at $N_f = 3$ the IR fixed point $a^* = 0.299$ for the gluon bremsstrahlung coupling, smaller then the corresponding value $a^* = 0.578$ for the pinch coupling which is subject to rather large uncertainties.

As a third alternative, I would like to suggest the "universal coupling" introduced in [12], because of its simplicity. It is defined[3] by the condition $\delta_i = 0$ for all i's, i.e. $a^*(\epsilon) \equiv \epsilon$, and therefore its beta function can be expressed entirely in term of the critical exponent

$$\begin{aligned}\beta(a, \epsilon) &= (a - \epsilon)\gamma(a) \\ &= -(\beta_0\ a^2 + \beta_{1,0}a^3)(1 + g_1 a + g_2 a^2 + \cdots)\,.\end{aligned} \tag{4.3}$$

At $N_f = 3$ the IR fixed point is $a^* = \epsilon = 0.336$. The scale is fixed knowing that $\frac{\beta_{2,0}}{\beta_{1,0}}|\text{universal} = \beta_{1,1}$ (from $\delta_1 = 0$), which determines (eq. (4.2)) $d_1|\text{universal} = -1.137$, and the natural assumption that the term proportional to β_0 in eq. (4.1) is the same.

The bremsstrahlung coupling beta function at $N_f = 3$ is shown in figure 1. Note the *negative* ultraviolet fixed point at $a \simeq -0.17$. It corresponds to a zero of the critical exponent eq. (3.5) at $\epsilon \simeq -0.15$, and is a necessary condition for the scenario in [6] to determine the bottom N_f^* of the conformal window from the condition $\gamma(\epsilon) = 1$, which yields $N_f^* \simeq 4$ if one uses eq. (3.5). The resulting running coupling is shown in figure 2, where I used $\alpha_s^{\overline{MS}}(M_Z) = 0.117$ as input (eq. (4.1) yields the corresponding input value of the bremsstrahlung coupling).

[3] I assume linear N_f dependence. Otherwise there is the more general solution $\beta(a, \epsilon) = \gamma(a)(a - \epsilon) + \gamma_1(a)(a - \epsilon)^2 + \cdots$.

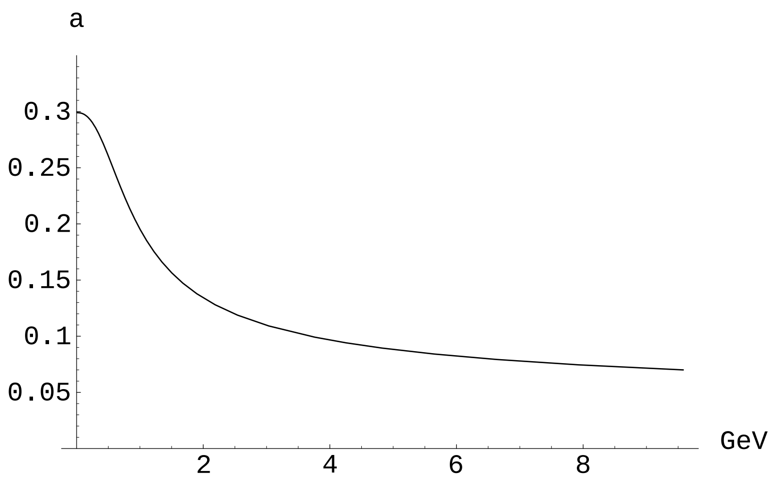

Figure 2: The bremsstrahlung coupling ($N_f = 3$, $\alpha_s^{\overline{MS}}(M_Z) = 0.117$).

	bremsstrahlung	universal	pinch
a_0	0.207 (0.222)	0.225 (0.246)	0.330 (0.366)
a_1	0.176 (0.198)	0.187 (0.212)	0.256 (0.300)
a_3	0.155 (0.173)	0.163 (0.183)	0.210 (0.243)

Table 1: Moments for $\alpha_s^{\overline{MS}}(M_Z) = 0.117$.

	bremsstrahlung	universal	pinch
a_0	0.217 (0.232)	0.237 (0.259)	0.353 (0.390)
a_1	0.188 (0.213)	0.202 (0.230)	0.281 (0.332)
a_3	0.167 (0.189)	0.176 (0.201)	0.233 (0.273)

Table 2: Moments for $\alpha_s^{\overline{MS}}(M_Z) = 0.120$.

It is then straightforward to compute the first few low energy moments of the coupling

$$a_{2n-1}(\mu_I) \equiv \int_0^{\mu_I^2} n \frac{dk^2}{k^2} \left(\frac{k^2}{\mu_I^2}\right)^n a(k^2)\,. \tag{4.4}$$

In term of the beta function they are given by

$$a_{2n-1}(\mu_I) = \int_{a^*}^{a_I} n \frac{da}{\beta(a)} a \exp[n\Phi(a, a_I)]\,, \tag{4.5}$$

where

$$\Phi(a, a_I) \equiv \int_{a_I}^{a} \frac{dx}{\beta(x)} = \log\left(\frac{k^2}{\mu_I^2}\right) \tag{4.6}$$

is the solution of the renormalization group equation with $a_I \equiv a(\mu_I^2)$. Taking $\mu_I = 2GeV$, one gets the results in table 1 if $\alpha_s^{\overline{MS}}(M_Z) = 0.117$ and those in table 2 if $\alpha_s^{\overline{MS}}(M_Z) = 0.120$. Note the sensitivity to the high energy input value of α_s.

These results are subjected to theoretical uncertainties, stemming from the magnitude of the IR values of the coupling which should induce seizable higher order corrections. The convergence of the Banks-Zaks expansion is bad in the pinch coupling case (which has a large IR value), and the knowledge of the 3-loop beta function coefficient and of δ_2 is essential for a more reliable prediction. The situation looks better for the"universal coupling"and the bremsstrahlung coupling. To assess the convergence of the expansion, the results for the moments in the NLO approximation where one uses only the first two terms in the Banks-Zaks expansion of γ (eq. (3.5)) are quoted within parenthesis in the tables.

The $n = 0$ moment gives the process-independent part of the normalization of the $1/Q$ power corrections. If one uses the gluon bremsstrahlung ansatz for the coupling, the predicted value is in qualitative agreement with the experimentally determined [5] one ($a_0 \simeq 0.14 - 0.17$), although it should be remembered that the latter depends on the way the "perturbative part" of the amplitude (the piece above μ_I in eq. (2.10)) is handled, as well as upon extra assumptions in the case of non-inclusive minkowskian observables. The $n = 3$ moment gives the normalization of the "gluon condensate"

$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\mu_I} = \frac{3}{2\pi^2} a_3(\mu_I)\mu_I^4 \,. \tag{4.7}$$

Note the definition used here involves an arbitrary IR cut-off μ_I, as necessary in the case of renormalon-related power corrections. If one wants to compare[4] to the effective phenomenological SVZ definition [1], one can just compute the integral in eq. (2.4) (which does not depend on μ_I) for any given euclidean observable where the gluon condensate gives the leading power correction, and fit the result with the SVZ ansatz. For instance, for the Adler D-function

$$D(Q^2) \simeq a(Q^2) + \frac{2\pi^2}{3}\frac{1}{Q^4}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle , \tag{4.8}$$

where the $2\pi^2/3$ factor is the leading order coefficient function. Similarly, the SVZ condensate $\langle \frac{\alpha_s}{\pi} G^2 \rangle$ could be defined from the basic observable $a_3(\mu_I)$ (eq. (4.4) with $n = 2$), where the IR cut-off μ_I now plays the role of the high energy scale Q, by

$$a_3(\mu_I) \simeq a(\mu_I^2) + \frac{2\pi^2}{3}\frac{1}{\mu_I^4}\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle . \tag{4.9}$$

For $\mu_I = 2GeV$, eq. (4.9) yields $\langle \frac{\alpha_s}{\pi} G^2 \rangle \simeq 0.05\ GeV^4$ for $\alpha_s^{\overline{MS}}(M_Z) = 0.117$, which looks reasonable compared to the standard SVZ value. However, this comparison is actually devoid of significance due to the following intriguing fact: varying the scale μ_I in eq. (4.9), one finds the discrepancy $a_3(\mu_I) - a(\mu_I^2)$ between $a_3(\mu_I)$ and its lowest order perturbative approximation $a(\mu_I^2)$ decreases *much slower* then the

[4] I am indebted to Al. Mueller for raising the question.

inverse fourth power of μ_I! A similar result is obtained if one uses eq. (4.8) (with the "loop momentum distribution function" $\Phi(k^2/Q^2)$ taken from [9]). Since the (principal value regulated) Borel sum of the perturbative series associated to the observables $a_3(\mu_I)$ (or $D(Q^2)$) are known [22, 23] to differ from the exact values by just such an $\mathcal{O}(1/\mu_I^4)$ (resp. $\mathcal{O}(1/Q^4)$) correction, one is bound to conclude that the naive treatment of approximating the Borel sum by its leading order term does not work[5] here. This is another point of discrepancy with the standard SVZ procedure, on top of the assumption that the "perturbative part" of the condensate dominates.

5. Discussion and conclusions

The essential assumption in the present approach is that the *perturbative* beta function has an IR fixed point at least down to $N_f = 3$. This is partly implemented by constructing beta functions with *negative* three-loop coefficients: at NNLO the method of section 3 yields $\beta_2 = \beta_{2,0} + \beta_{2,1}\ \beta_0$ where both $\beta_{2,0}$ and $\beta_{2,1}$ turn out negative (see footnote 6) for the considered couplings. Actually, essentially the same results can be obtained (at least for the bremsstrahlung coupling) in a simpler way, which makes it transparent the reason for the existence of the IR fixed point. Indeed, consider the 4-loop beta function eq. (3.1), and observe that in the IR region the usual power counting should be modified: namely, given that a is $\mathcal{O}(\beta_0)$ there, to $\mathcal{O}(a^6)$ accuracy one should drop the $\mathcal{O}(\beta_0^2)$ term in β_2, and keep only the leading $\mathcal{O}((\beta_0)^0)$ term in β_3, i.e. use the effective 4-loop beta function (in accordance with the remark in footnote 2)

$$\beta_{\text{eff}}(a,\epsilon) = -\beta_0 a^2 - \beta_1 a^3 - (\beta_{2,0} + \beta_{2,1}\beta_0)a^4 - \beta_{3,0}\ a^5 + \mathcal{O}(a^6) \tag{5.1}$$

(in the ultraviolet region, this beta function has of course only the $\mathcal{O}(a^4)$ accuracy of the 2-loop beta function). In the case of the bremsstrahlung coupling, the results obtained using the 4-loop β_{eff} turn out to be very close to those of section 4 in the NNLO approximation. For instance β_{eff} has an IR fixed point at $a^* = 0.294$ if $N_f = 3$ (I used $\beta_{3,0} = 37.76$ from eq. (3.6)), and one gets: $a_0 = 0.201$, $a_1 = 0.177$ and $a_3 = 0.156$ if $\alpha_s^{\overline{MS}}(M_Z) = 0.117$, and $a_0 = 0.210$, $a_1 = 0.189$ and $a_3 = 0.168$ if $\alpha_s^{\overline{MS}}(M_Z) = 0.120$. Similarly in NLO one should use a 3-loop $\beta_{\text{eff}}(a,\epsilon) = -\beta_0\ a^2 - \beta_1\ a^3 - \beta_{2,0}\ a^4 + \mathcal{O}(a^5)$, and in LO a 2-loop $\beta_{\text{eff}}(a,\epsilon) = -\beta_0\ a^2 - \beta_{1,0}\ a^3 + \mathcal{O}(a^4)$. The presence of an IR fixed point in β_{eff} down to low values of N_f seems to be a general phenomenon, at least up to NLO. This is obvious in LO, since $\beta_{1,0}$ is scheme independent, but less so in NLO where $\beta_{2,0}$ is scheme dependent. Nevertheless it turns out that $\beta_{2,0}$ is *negative* for all known physical effective charges [18], as well as for the three couplings quoted above. Consequently, there may be a positive zero in the 3-loop β_{eff} correctly signalling an IR fixed point, even if the standard 3-loop beta function has no positive zero with all its coefficients of the same sign.

[5]Similar results are obtained if one uses [8] the BLM scale [18] in a.

At NNLO, the presence of an IR fixed point in the 4-loop $\beta_{\rm eff}$ may be jeopardized by large positive values of $\beta_{2,1}$ and (or) $\beta_{3,0}$. Actually, $\beta_{2,1}$ turns out to be negative for all known[6] effective charges (except the one ("a_V") defined by the static QCD potential, where it is positive [18] but small enough not to destabilize the fixed point). The real problem comes from the 4-loop coefficient $\beta_{3,0}$, which is *positive* for all known effective charges (except again a_V, where it is negative and tiny). In the case of the pinch coupling, it turns out in fact too large (one gets $\beta_{3,0} = 164.7$ from eq. (3.6)) for the 4-loop $\beta_{\rm eff}$ to have an IR fixed point if $N_f < 13$. Similarly, in the case of the Adler D-function effective charge where $\beta_{3,0} = 127$, the 4-loop $\beta_{\rm eff}$ does not have an IR fixed point if $N_f < 11$. For all other effective charges however the 4-loop $\beta_{\rm eff}$ does exhibit an IR fixed point down to $N_f = 0$! However, in those cases of large positive $\beta_{3,0}$ (which is a consequence of a *small* $\beta_{2,0}$, see eq. (3.6)), the method of section 3 provides an effective resummation of the relevant higher order terms, obtained under the assumption the Banks-Zaks expansions of the IR fixed point and of the critical exponent do converge: *all* known effective charges then appear[7] after resummation to have an IR fixed point down to $N_f = 0$ (although the convergence of the fixed point Banks-Zaks expansion becomes problematic already at $N_f = 3$ for some of them, such as the pinch coupling).

The suggestion of perturbative freezing of the coupling at low N_f was first made in [13]. There is however an essential difference with the present proposal: it is *not* suggested here that the perturbative IR fixed point has anything to do with the low energy behavior of the full QCD amplitudes below the conformal window, which is entirely non-perturbative. For instance, as observed in [25] spontaneous chiral symmetry breaking considerations at large N_c imply the Adler D-function must vanish at zero momentum, which is inconsistent with the positive value expected from perturbative freezing. What is suggested instead is that the perturbative freezing at low N_f is relevant to determine the normalization of renormalon-related condensates and power corrections which appear in the *short distance* expansion of amplitudes. This amounts to the recognition that objects like the "gluon condensate", at the difference of the quark condensate, are of a basically "perturbative" nature, and thus

[6] For the pinch coupling and the bremsstrahlung coupling, $\beta_{2,1}$ has been "predicted" from the assumption that $\delta_2 \simeq 0$, which yields (eq. (3.3)) $\beta_{2,1} = -23.6$ for the pinch coupling and $\beta_{2,1} = -7.43$ for the bremsstrahlung coupling. This assumption turns out to yield rather good results in the case of the effective charges associated to the Adler D-function and the polarized (g_1) and non-polarized (F_1) Bjorken sum rules, for which the "predicted" values are respectively $\beta_{2,1} = -16.17, -9.98, -6.97$ compared to the exact values (corrected for some numerical inaccuracies in [18]) $\beta_{2,1} = -15.94, -11.19, -6.81$. The partial reason for this success are the large cancellations between g_2 (the "scheme independent" contribution to δ_2 in eq. (3.3)) and the "scheme dependent" contribution which involves δ_1 and $\beta_{2,1}$.

[7] This is even true for the effective charge associated [24] to Higgs decay. In this case however one gets a large fixed point value $a^* = \mathcal{O}(1)$, and convergence of the Banks-Zaks expansion is doubtful for $N_f < 4$.

unrelated to "genuine" non-perturbative properties of the vacuum such as chiral symmetry breaking or confinement. The notion of a "conformal window" is an essential part of the proposed scheme: only those power corrections which are already present within the conformal window are amenable to a perturbative treatment, and below the conformal window there are other really "non-perturbative" contributions which are crucial to determine the true low energy properties of QCD. Moreover, it was shown in [6] that the assumption that the perturbative IR fixed point persists below the bottom N_f^* of the conformal window leads to the condition $\gamma(\epsilon) = 1$ to determine N_f^*. It is interesting that this condition gives $N_f^* \simeq 4$, rather close to the "real life" QCD value $N_f = 3$, which might give an alternative justification to the suggested calculation procedure based on the "anti-SVZ" hypothesis that the "perturbative" piece of the condensates actually dominates over the "non-perturbative" fluctuations. Note that the opposite SVZ assumption of dominance of the non-perturbative piece has been questioned previously in the literature (see e.g. [2]). Furthermore, even if the present assumption turns out to be invalid, the results of this paper are still useful to extract from experiment the "truly non-perturbative" part C_{NP}, which is given a completely unambiguous definition through eq. (2.9).

The typical example of a "perturbative" conformal window amplitude is the "single dressed gluon" integral of eq. (2.4), where the running coupling inside the integral is IR finite, and calculable from *perturbative* input through an (eventually resummed) Banks-Zaks expansion. It is implicitly assumed that this particular coupling is free of IR renormalons and can be unambiguously determined from its perturbative series (say, by Borel summation). The corresponding Banks-Zaks series should then be also Borel summable. Note also that the integral eq. (2.4) is free of any renormalon ambiguity, although renormalons are present in the corresponding perturbative series, but is still expected to be affected below the conformal window by unphysical Landau singularities in the complex Q^2 plane. Such an amplitude [22, 23] represents a natural form of a generalized perturbation theory, which gives the background on top of which genuine non-perturbative contributions may take place below N_f^*. The calculation of the "perturbative condensate", although using only perturbative information, goes beyond a mere renormalon estimate, since there is usually a part [26, 6] of the low momentum contribution of the perturbative coupling which is *not* determined *only* [26, 27] by renormalons. The assumption that renormalon-related power corrections are "perturbative" in the above sense also gives a straightforward justification to the IR finite coupling approach to power corrections, and leaves no arbitrary free parameter (except of course the overall QCD scale) to be fixed from experiment. The main conceptual problem in this framework remains to find the identity of the (hopefully unique) perturbative IR finite QCD coupling which determines the power corrections, and derive the systematic form to all orders of the (yet to be constructed) generalized perturbation theory, perhaps [18] along the lines of a QCD "skeleton expansion": this is however a problem of a basically perturbative nature.

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